

# A FINITE DIMENSIONAL $A_\infty$ ALGEBRA EXAMPLE

MICHAEL P. ALLOCCA AND TOM LADA

*Dedicated to Tornike Kadeishvili on the occasion of his 60th birthday*

ABSTRACT. We construct an example of an  $A_\infty$  algebra structure defined over a finite dimensional graded vector space.

## INTRODUCTION

$A_\infty$  algebras (or sha algebras) and  $L_\infty$  (or sh Lie algebras) have been topics of current research. Construction of small examples of these algebras can play a role in gaining insight into deeper properties of these structures. These examples may prove useful in developing a deformation theory as well as a representation theory for these algebras.

In [2], an  $L_\infty$  algebra structure on the graded vector space  $V = V_0 \oplus V_1$  where  $V_0$  is a 2 dimensional vector space, and  $V_1$  is a 1 dimensional space, is discussed. This surprisingly rich structure on this small graded vector space was shown by Kadeishvili and Lada, [3], to be an example of an open-closed homotopy algebra (OCHA) defined by Kajiura and Stasheff [4]. In an unpublished note [1] M. Daily constructs a variety of other  $L_\infty$  algebra structures on this same vector space.

In this article we add to this collection of structures on the vector space  $V$  by providing a detailed construction of non-trivial  $A_\infty$  algebra data for  $V$ .

## 1. $A_\infty$ ALGEBRAS

We first recall the definition of an  $A_\infty$  algebra (Stasheff [6]).

**Definition 1.1.** *Let  $V$  be a graded vector space. An  $A_\infty$  structure on  $V$  is a collection of linear maps  $m_k : V^{\otimes k} \rightarrow V$  of degree  $2 - k$  that satisfy the identity*

$$\sum_{\lambda=0}^{n-1} \sum_{k=1}^{n-\lambda} \alpha m_{n-k+1}(x_1 \otimes \cdots \otimes x_\lambda \otimes m_k(x_{\lambda+1} \otimes \cdots \otimes x_{\lambda+k}) \otimes x_{\lambda+k+1} \otimes \cdots \otimes x_n) = 0$$

where  $\alpha = (-1)^{k+\lambda+k\lambda+kn+k(|x_1|+\cdots+|x_\lambda|)}$ , for all  $n \geq 1$ .

This utilizes the cochain complex convention. One may alternatively utilize the chain complex convention by requiring each map  $m_k$  to have degree  $k - 2$ .

We will define the desuspension of  $V$  (denoted  $\downarrow V$ ) as the graded vector space with indices given by  $(\downarrow V)_n = V_{n+1}$ , and the desuspension operator,  $\downarrow: V \rightarrow (\downarrow V)$  (resp. suspension operator  $\uparrow: (\downarrow V) \rightarrow V$ ) in the natural sense. We will also employ the usual Koszul sign convention in this setting: whenever two symbols (objects or maps) of degree  $p$  and  $q$  are commuted, a factor of  $(-1)^{pq}$  is introduced. Subsequently,  $\uparrow^{\otimes n} \circ \downarrow^{\otimes n} = (-1)^{\frac{n(n-1)}{2}} id$  and  $\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n = (-1)^{\sum_{i=1}^n (n-i)|x_i|} \downarrow^{\otimes n}(x_1 \otimes x_2 \otimes \cdots \otimes x_n)$ .

Stasheff also showed that an  $A_\infty$  structure on  $V$  is equivalent to the existence of a degree 1 coderivation  $D: T^*(\downarrow V) \rightarrow T^*(\downarrow V)$  with the property  $D^2 = 0$ . Here,  $T^*(\downarrow V)$  is the tensor coalgebra on the graded vector space  $\downarrow V$ . Such a coderivation is constructed by defining  $m'_k: (\downarrow V^{\otimes k}) \rightarrow \downarrow V$  by  $m'_k = (-1)^{\frac{k(k-1)}{2}} \downarrow \circ m_k \circ \uparrow^{\otimes k}$  and then extending each  $m'_k$  to a coderivation on  $T^*(\downarrow V)$ . By “abuse of notation”,  $m'_k$  can be described by

$$\begin{aligned} m'_k(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) &= \sum_{i=0}^{n-1} (1^{\otimes i} \otimes m'_k \otimes 1^{\otimes n-i-1})(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) \\ &= \sum_{i=0}^{n-1} (-1)^{(k-2)(|x_1|+\cdots+|x_i|-i)} (\downarrow x_1 \otimes \cdots \otimes \downarrow x_i \otimes m'_k(\downarrow x_{i+1} \otimes \cdots \otimes \downarrow x_{i+k}) \otimes \downarrow x_{i+k+1} \otimes \cdots \otimes \downarrow x_n) \end{aligned}$$

We then define  $D := \sum_{k=1}^{\infty} m'_k$ .

## 2. A FINITE DIMENSIONAL EXAMPLE

Let  $V$  denote the graded vector space given by  $V = \bigoplus V_n$  where  $V_0$  has basis  $\langle v_1, v_2 \rangle$ ,  $V_1$  has basis  $\langle w \rangle$ , and  $V_n = 0$  for  $n \neq 0, 1$ . Define a structure on  $V$  by the following linear maps  $m_n: V^{\otimes n} \rightarrow V$ :

$$\begin{aligned} m_1(v_1) &= m_1(v_2) = w \\ \text{For } n \geq 2: \quad m_n(v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{\otimes (n-2)-k}) &= (-1)^k s_n v_1, \quad 0 \leq k \leq n-2 \\ m_n(v_1 \otimes w^{\otimes (n-2)} \otimes v_2) &= s_{n+1} v_1 \\ m_n(v_1 \otimes w^{\otimes (n-1)}) &= s_{n+1} w \end{aligned}$$

where  $s_n = (-1)^{\frac{(n+1)(n+2)}{2}}$ , and  $m_n = 0$  when evaluated on any element of  $V^{\otimes n}$  that is not listed above. It is worth noting that this assumes the cochain convention regarding  $A_\infty$  algebra structures. Hence  $|v_1| = |v_2| = 0$  and  $|w| = 1$ .

**Theorem 2.1.** *The maps defined above give the graded vector space  $V$  an  $A_\infty$  algebra structure.*

The proof of this theorem relies on two lemmas:

**Lemma 2.2.** *Let  $m'_n := (-1)^{\frac{n(n-1)}{2}} \downarrow \circ m_n \circ \uparrow^{\otimes n}: (\downarrow V)^{\otimes n} \rightarrow \downarrow V$ . Under the preceding definitions for  $m_n$  and  $V$ , we obtain the following formulas for  $m'_n$ :*

$$\begin{aligned} m'_1 &= \downarrow m_1 \\ \text{For } n \geq 2: \quad m'_n(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes (n-2)-k}) &= \downarrow v_1, \quad 0 \leq k \leq n-2 \\ m'_n(\downarrow v_1 \otimes (\downarrow w)^{\otimes (n-2)} \otimes \downarrow v_2) &= \downarrow v_1 \\ m'_n(\downarrow v_1 \otimes (\downarrow w)^{\otimes (n-1)}) &= \downarrow w \end{aligned}$$

**Remark 2.3.** *Each  $m'_n$  is of degree 1.*

*Proof of Lemma 2.2.*  $m'_1(\downarrow x) = (-1)^0 \downarrow \circ m_1 \circ \uparrow(\downarrow x) = \downarrow m_1(x)$  for any  $x$ .

Now let  $n \geq 2$ . The majority of the work here is centered around computing the signs associated with the graded setting. The elements  $x_i$  and the maps  $\uparrow$ ,  $\downarrow$ , and  $m_n$  all contribute to an overall sign via their degrees. Observing these signs, we find

$$\begin{aligned} m'_n(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) &= (-1)^{\frac{n(n-1)}{2}} \downarrow \circ m_n \circ \uparrow^{\otimes n}(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) \\ &= \begin{cases} (-1)^{\sum_{i=1}^{n/2} |x_{2i-1}|} \downarrow m_n(x_1 \otimes x_2 \otimes \cdots \otimes x_n) & \text{if } n \text{ is even.} \\ (-1)^{\sum_{i=1}^{(n-1)/2} |x_{2i}|} \downarrow m_n(x_1 \otimes x_2 \otimes \cdots \otimes x_n) & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

First consider  $m'_n(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes (n-2)-k})$ ,  $0 \leq k \leq n-2$ . This computation may be divided into 4 cases based on the parity of  $n$  and  $k$ . If  $n$  and  $k$  are both even, then:

$$\begin{aligned} m'_n(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes (n-2)-k}) &= (-1)^{|v_1| + (\frac{n}{2}-1)|w|} \downarrow m_n(v_1 \otimes w^{\otimes k} \otimes v_1 \otimes w^{\otimes (n-2)-k}) \\ &= (-1)^{0 + \frac{n}{2}-1} (-1)^k s_n \downarrow v_1 \\ &= (-1)^{\frac{n}{2}-1} (-1)^{\frac{(n+1)(n+2)}{2}} \downarrow v_1 \\ &= (-1)^{\frac{n}{2}-1} (-1)^{(n+1)(\frac{n}{2}+1)} \downarrow v_1 \quad (*) \end{aligned}$$

If  $\frac{n}{2}$  is even, then  $(*) = (-1)^{\text{odd}} (-1)^{\text{odd} * \text{odd}} \downarrow v_1 = \downarrow v_1$  where ‘odd’ denotes an odd number.

If  $\frac{n}{2}$  is odd, then  $(*) = (-1)^{\text{even}} (-1)^{\text{odd} * \text{even}} \downarrow v_1 = \downarrow v_1$  where ‘even’ denotes an even number.

A similar argument holds in the remaining 3 cases. Hence

$$m'_n(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes (n-2)-k}) = \downarrow v_1, \quad 0 \leq k \leq n-2$$

Now consider  $m'_n(\downarrow v_1 \otimes (\downarrow w)^{\otimes (n-2)} \otimes \downarrow v_2)$ . This computation may be divided into 2 cases based on the parity of  $n$ . If  $n$  is even, then:

$$\begin{aligned}
m'_n(\downarrow v_1 \otimes (\downarrow w)^{\otimes(n-2)} \otimes \downarrow v_2) &= (-1)^{|v_1| + (\frac{n}{2}-1)|w|} m_n(\downarrow v_1 \otimes (\downarrow w)^{\otimes(n-2)} \otimes \downarrow v_2) \\
&= (-1)^{\frac{n}{2}-1} s_{n+1} \downarrow v_1 \\
&= (-1)^{\frac{n}{2}-1} (-1)^{\frac{(n+2)(n+3)}{2}} \downarrow v_1 \\
&= (-1)^{\frac{n}{2}-1} (-1)^{(\frac{n}{2}-1)(n+3)} \downarrow v_1 \quad (*)
\end{aligned}$$

If  $\frac{n}{2}$  is even, then  $(*) = (-1)^{\text{odd}} (-1)^{\text{odd} * \text{odd}} \downarrow v_1 = \downarrow v_1$  where ‘odd’ denotes an odd number.

If  $\frac{n}{2}$  is odd, then  $(*) = (-1)^{\text{even}} (-1)^{\text{even} * \text{odd}} \downarrow v_1 = \downarrow v_1$  where ‘even’ denotes an even number.

A similar argument holds in the case that  $n$  is odd. Hence

$$m'_n(\downarrow v_1 \otimes (\downarrow w)^{\otimes(n-2)} \otimes \downarrow v_2) = \downarrow v_1$$

The preceding argument may also be repeated for  $m'_n(\downarrow v_1 \otimes (\downarrow w)^{\otimes(n-1)})$ . Hence

$$m'_n(\downarrow v_1 \otimes (\downarrow w)^{\otimes(n-1)}) = \downarrow w$$

□

**Lemma 2.4.** *Let  $D = \sum_{k=1}^{\infty} m'_k$  where  $m'_k$  is defined in Lemma 2.2. Let  $n \geq 2$  be a positive integer. Suppose  $D^2(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_m) = 0 \ \forall \ x_i \in V, \ 1 \leq m \leq n-1$ .*

$$\text{Then } D^2(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) = \sum_{i+j=n+1} m'_i m'_j (\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n)$$

*Proof.* We first note that

$$D^2(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) = \sum_{i+j \leq n+1} m'_i m'_j (\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n)$$

since  $m'_k(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_l) = 0$  for  $k > l$ . So

$$\begin{aligned}
D^2(\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) &= \sum_{i+j \leq n} m'_i m'_j (\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) \\
&\quad + \sum_{i+j=n+1} m'_i m'_j (\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n)
\end{aligned}$$

Hence it suffices to show that  $\sum_{i+j \leq n} m'_i m'_j (\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n) = 0$

Consider  $\sum_{i+j \leq n} m'_i m'_j (\downarrow x_1 \otimes \downarrow x_2 \otimes \cdots \otimes \downarrow x_n)$ : Since  $i+j \leq n$ , we can break this sum up into 4 different types of elements in  $(\downarrow V)^{\otimes k}$  based on whether the first and last terms in the tensor product contain  $m'_i$  or  $m'_j$ :

- Type 1: Elements with first term  $\downarrow x_1$  and last term  $\downarrow x_n$   
(example:  $\downarrow x_1 \otimes \downarrow x_2 \otimes m'_1(\downarrow x_3) \otimes m'_2(\downarrow x_4 \otimes \downarrow x_5) \otimes \downarrow x_6$ )
- Type 2: Elements with first term  $\downarrow x_1$  and last term containing  $m'_k$  for some  $k$   
(example:  $\downarrow x_1 \otimes \downarrow x_2 \otimes m'_3(\downarrow x_3 \otimes m'_2(\downarrow x_4 \otimes \downarrow x_5) \otimes \downarrow x_6)$ )
- Type 3: Elements with first term containing  $m'_k$  for some  $k$  and last term  $\downarrow x_n$   
(example:  $m'_2(\downarrow x_1 \otimes \downarrow x_2) \otimes m'_1(\downarrow x_3) \otimes \downarrow x_4 \otimes \downarrow x_5 \otimes \downarrow x_6$ )
- Type 4: Elements with first term containing  $m'_k$  and last term containing  $m'_l$  for some  $k, l$   
(example:  $m'_2(\downarrow x_1 \otimes \downarrow x_2) \otimes \downarrow x_3 \otimes \downarrow x_4 \otimes m'_2(\downarrow x_5 \otimes \downarrow x_6)$ )

Now each term of type 1 must be produced by  $m'_i m'_j$  with  $i + j \leq n - 1$ . Hence, by factorization of tensor products, all possible terms of type 1 are given by:

$$\begin{aligned}
 & (-1)^{2|x_1|-2} \left( \downarrow x_1 \otimes \left( \sum_{i+j \leq n-1} m'_i m'_j (\downarrow x_2 \otimes \downarrow x_3 \otimes \cdots \otimes \downarrow x_{n-1}) \right) \right) \otimes \downarrow x_n \\
 &= \left( \downarrow x_1 \otimes (D^2(\downarrow x_2 \otimes \downarrow x_3 \otimes \cdots \otimes \downarrow x_{n-1})) \right) \otimes \downarrow x_n \\
 &= (\downarrow x_1 \otimes 0) \otimes \downarrow x_n \\
 &= 0
 \end{aligned}$$

since  $D^2 = 0$  when evaluated on  $n - 2$  terms. A similar argument holds for the type 2 and type 3 summands.

We now consider type 4 terms. Consider an arbitrary element of type 4:

$$m'_i(\downarrow x_1 \otimes \cdots \otimes \downarrow x_i) \otimes \downarrow x_{i+1} \otimes \cdots \otimes \downarrow x_{n-j} \otimes m'_j(\downarrow x_{n-j+1} \otimes \cdots \otimes \downarrow x_n)$$

Consider how this arbitrary element is generated: We begin with

$$m'_i m'_j (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)$$

We then apply  $m'_j$  to the last  $j$  terms, which yields:

$$(-1)^{|x_1| + \cdots + |x_{n-j}| - (n-j)} m'_i (\downarrow x_1 \otimes \cdots \otimes \downarrow x_{n-j} \otimes m'_j (\downarrow x_{n-j+1} \otimes \cdots \otimes \downarrow x_n))$$

Finally we apply  $m'_i$  to the first  $i$  terms:

$$(-1)^{|x_1| + \cdots + |x_{n-j}| - (n-j)} m'_i (\downarrow x_1 \otimes \cdots \otimes \downarrow x_i) \otimes \downarrow x_{i+1} \otimes \cdots \otimes \downarrow x_{n-j} \otimes m'_j (\downarrow x_{n-j+1} \otimes \cdots \otimes \downarrow x_n) \quad (*)$$

Each of these arbitrary type 4 elements can be paired up with an element generated by  $m'_j m'_i$  as follows: Begin with

$$m'_j m'_i (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)$$

Then apply  $m'_i$  to the first  $i$  terms:

$$m'_j (m'_i (\downarrow x_1 \otimes \cdots \otimes \downarrow x_i) \otimes \downarrow x_{i+1} \otimes \cdots \otimes \downarrow x_n)$$

Finally, apply  $m'_j$  to the last  $j$  terms:

$$(-1)^{|x_1|+\dots+|x_{n-j}|-(n-j)+1} m'_i(\downarrow x_1 \otimes \dots \otimes \downarrow x_i) \otimes \downarrow x_{i+1} \otimes \dots \otimes \downarrow x_{n-j} \otimes m'_j(\downarrow x_{n-j+1} \otimes \dots \otimes \downarrow x_n) \quad (**)$$

Since these type 4 elements were arbitrary, and  $(*) + (**) = 0$ , all type 4 terms added together equal 0. Hence, all type 1, 2, 3, and 4 terms yield 0, and so

$$\sum_{i+j \leq n} m'_i m'_j(\downarrow x_1 \otimes \downarrow x_2 \otimes \dots \otimes \downarrow x_n) = 0$$

□

*Proof of Theorem 2.1.* It is clear that each map  $m_n$  is of degree  $2 - n$ . To prove that these maps yield an  $A_\infty$  structure, one may verify that they satisfy the identity given in definition 1.1. However, this is a rather daunting task, due to the varying signs,  $s_n$ , accompanying the  $m_n$  maps. To utilize an alternative method of proof, we construct a degree 1 coderivation,  $D$ , as described in section 1.

In the context of Theorem 2.1, we may use the definition for  $m'_k$  given by Lemma 2.2 to construct  $D$ . It then suffices to show that  $D^2 = 0$ .

We aim to prove  $D^2 = 0$  by induction on the number of inputs for  $D$ . It is worth first noting that  $D = \sum_{k=1}^{\infty} m'_k$ , however  $D(\downarrow x_1 \otimes \dots \otimes \downarrow x_n) = \sum_{k=1}^n m'_k(\downarrow x_1 \otimes \dots \otimes \downarrow x_n)$  since  $m'_k(\downarrow x_1 \otimes \dots \otimes \downarrow x_n) = 0$  for  $k \geq n$ .

For  $n = 1$ , we have  $D^2(\downarrow x) = m'_1 m'_1(\downarrow x) = \downarrow m_1^2(x) = 0 \forall x \in V$ .

Now assume  $D^2(\downarrow x_1 \otimes \dots \otimes \downarrow x_{n-1}) = 0$ . We aim to show that  $D^2(\downarrow x_1 \otimes \dots \otimes \downarrow x_n) = 0$ :

**Remark 2.5.** Since  $m'_i$  and  $m'_j$  are linear, it is sufficient to show that  $D^2 = 0$  on only basis elements.

By Lemma 2.4,  $D^2(\downarrow x_1 \otimes \dots \otimes \downarrow x_n) = \sum_{i+j=n+1} m'_i m'_j(\downarrow x_1 \otimes \dots \otimes \downarrow x_n)$ , hence it suffices to show that  $\sum_{i+j=n+1} m'_i m'_j(\downarrow x_1 \otimes \dots \otimes \downarrow x_n) = 0, \forall x_1 \dots x_n \in V$ .

It is advantageous to approach this problem from the bottom up, since  $x_1 \dots x_n \in V$  implies calculating  $3^n$  different combinations of elements. That is, we consider only nontrivial (nonzero) elements in the sum  $\sum_{i+j=n+1} m'_i m'_j(\downarrow x_1 \otimes \dots \otimes \downarrow x_n)$ . Now since  $i + j = n + 1$ , we observe that  $m'_i m'_j(\downarrow x_1 \otimes \dots \otimes \downarrow x_n) \in (\downarrow V)^{\otimes 1}$ . Since, by definition,  $m'_i$  cannot produce

the element  $\downarrow v_2$ , the seemingly large task of considering nontrivial  $m'_i m'_j(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)$  yields only two possibilities:

$$\begin{aligned} m'_i m'_j(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) &= c \downarrow v_1 \\ \text{or } m'_i m'_j(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) &= c \downarrow w \text{ for some constant, } c. \end{aligned}$$

Therefore if  $m'_i m'_j(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) \neq 0$  for some  $i + j = n + 1$ , then  $\sum_{i+j=n+1} m'_i m'_j(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)$  is a sum of  $\downarrow v_1$ 's or  $\downarrow w$ 's.

We first consider the manner in which  $m'_i m'_j(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)$  yields a  $\downarrow w$ :

By definition of  $m'_n$ ,  $\downarrow w$  must be produced by  $m'_i(\downarrow v_1 \otimes (\downarrow w)^{\otimes(i-1)}) (*)$ . To accomplish this, the original arrangement  $\downarrow x_1 \otimes \cdots \otimes \downarrow x_n$  must satisfy  $x_1 = v_1$  and must contain exactly one more ' $v$ ' ( $v = v_1$  or  $v_2$ ).

• **Case 1:**  $v = v_1$ . Let us consider  $m'_i m'_j(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes(n-2)-k})$ ,  $0 \leq k \leq n-2$ . Now, to produce  $(*)$ ,  $m'_j$  must 'catch' (1) both  $\downarrow v_1$ 's, or (2) only the second  $\downarrow v_1$ .

(1) We have  $m'_j(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes(n-2)-k}) = \downarrow v_1$ ,  $k+2 \leq j \leq n$ . This yields  $m'_i(\downarrow v_1 \otimes (\downarrow w)^{\otimes(n-j)}) = \downarrow w$ . Now since  $k+2 \leq j \leq n$ , there are  $n - (k+2) + 1 = n - k - 1$  such terms in  $\sum_{i+j=n+1} m'_i m'_j(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes(n-2)-k})$ .

(2) We have  $(-1)^{|v_1|+k|w|-(k+1)} m'_i \left( \downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \left[ m'_j(\downarrow v_1 \otimes (\downarrow w)^{\otimes(j-1)}) \right] \otimes (\downarrow w)^{\otimes(n-2)-k-(j-1)} \right) = -\downarrow w$ ,  $1 \leq j \leq n - k - 1$ . Similarly, there are  $(n - k - 1) - 1 + 1 = n - k - 1$  such terms in  $\sum_{i+j=n+1} m'_i m'_j(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes(n-2)-k})$ .

$$\Rightarrow \sum_{i+j=n+1} m'_i m'_j(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes(n-2)-k}) = (n-k-1) \downarrow w - (n-k-1) \downarrow w = 0.$$

• **Case 2:**  $v = v_2$ . Let us consider  $m'_i m'_j(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_2 \otimes (\downarrow w)^{\otimes(n-2)-k})$ ,  $0 \leq k \leq n-2$ . Similarly, to produce  $(*)$ ,  $m'_j$  must 'catch' (1) both  $\downarrow v_1$  and  $\downarrow v_2$ , or (2) only  $\downarrow v_2$ .

For (1), the only nontrivial way to do this yields:

$$m'_{n-k-1}(m'_{k+2}(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_2) \otimes (\downarrow w)^{\otimes(n-2)-k}) = \downarrow w$$

and for (2), the only nontrivial way to do this yields:

$$\begin{aligned} & (-1)^{|v_1|+k|w|-(k+1)} m'_n(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes m'_1(\downarrow v_2) \otimes (\downarrow w)^{\otimes (n-2)-k}) = -\downarrow w \\ \Rightarrow \sum_{i+j=n+1} m'_i m'_j(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes (n-2)-k}) &= \downarrow w - \downarrow w = 0. \end{aligned}$$

In either case, if  $m'_i m'_j(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)$  produces  $\downarrow w$ 's, then

$$\sum_{i+j=n+1} m'_i m'_j(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) = 0.$$

We now consider the manner in which  $m'_i m'_j(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n)$  yields a  $\downarrow v_1$ :

By definition of  $m'_n$ ,  $\downarrow v_1$  must be produced by either  $m'_i(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes (i-2)-k})$  or  $m'_i(\downarrow v_1 \otimes (\downarrow w)^{\otimes (i-2)} \otimes \downarrow v_2)$ .

• **Case 1:**  $\downarrow v_1$  is produced by  $m'_i(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes (i-2)-k})$ .

We examine the 4 different possibilities for which  $m'_j$  can yield this arrangement:

- (i)  $m'_j$  produces the first  $\downarrow v_1$ . (ii)  $m'_j$  produces a  $\downarrow w$  in  $(\downarrow w)^{\otimes k}$ .
- (iii)  $m'_j$  produces the second  $\downarrow v_1$ . (iv)  $m'_j$  produces a  $\downarrow w$  in  $(\downarrow w)^{\otimes (i-2)-k}$ .

A key observation to make here is that (i), (ii), (iii), and (iv) imply that the original arrangement  $\downarrow x_1 \otimes \cdots \otimes \downarrow x_n$  must contain exactly 3  $v$ 's, once again with  $x_1 = v_1$ . This yields 4 subcases:

- *Subcase 1:* We have  $m'_i m'_j(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes l} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes n-k-l-3})$ :
- *Subcase 2:* We have  $m'_i m'_j(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes l} \otimes \downarrow v_2 \otimes (\downarrow w)^{\otimes n-k-l-3})$ :
- *Subcase 3:* We have  $m'_i m'_j(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_2 \otimes (\downarrow w)^{\otimes l} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes n-k-l-3})$ :
- *Subcase 4:* We have  $m'_i m'_j(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_2 \otimes (\downarrow w)^{\otimes l} \otimes \downarrow v_2 \otimes (\downarrow w)^{\otimes n-k-l-3})$ :

Let us consider subcase 1:

(i)  $m'_j$  must take the first two  $\downarrow v_1$ 's. We have:

$$m'_i \left( \left[ m'_j(\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes j-k-2}) \otimes (\downarrow w)^{\otimes l-(j-k-2)} \right] \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes n-k-l-3} \right) = \downarrow v_1$$

Now  $k+2 \leq j \leq l+k+2$ , so there are  $(l+k+2) - (k+2) + 1 = l+1$  such terms.

(ii)  $m'_j$  must take only the second  $\downarrow v_1$ . We have:

$$(-1)^{|v_1|+k|w|-(k+1)} m'_i \left( \downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \left[ m'_j (\downarrow v_1 \otimes (\downarrow w)^{\otimes (j-1)}) \otimes (\downarrow w)^{\otimes l-(j-1)} \right] \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes n-k-l-3} \right) = -\downarrow v_1$$

Now  $1 \leq j \leq l+1$ , so there are  $(l+1) - 1 + 1 = l+1$  such terms.

(iii)  $m'_j$  must take the second and third  $\downarrow v_1$ 's. We have:

$$(-1)^{|v_1|+k|w|-(k+1)} m'_i \left( \downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \left[ m'_j (\downarrow v_1 \otimes (\downarrow w)^{\otimes l} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes j-l-2}) \right] \otimes (\downarrow w)^{\otimes n-k-j+1} \right) = -\downarrow v_1$$

Now  $l+2 \leq j \leq n-k-1$ , so there are  $(n-k-1) - (l+2) + 1 = n-k-l-2$  such terms.

(iv)  $m'_j$  must take only the third  $\downarrow v_1$ . We have:

$$(-1)^{2|v_1|+(k+l)|w|-(k+l+2)} m'_i \left( \downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes l} \otimes \left[ m'_j (\downarrow v_1 \otimes (\downarrow w)^{\otimes (j-1)}) \otimes (\downarrow w)^{\otimes n-k-l-j-2} \right] \right) = \downarrow v_1$$

Now  $1 \leq j \leq n-k-l-2$ , so there are  $(n-k-l-2) - 1 + 1 = n-k-l-2$  such terms.

$$\begin{aligned} &\Rightarrow \sum_{i+j=n+1} m'_i m'_j (\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes l} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes n-k-l-3}) = (l+1) \downarrow v_1 - (l+1) \downarrow \\ &v_1 - (n-k-l-2) \downarrow v_1 + (n-k-l-2) \downarrow v_1 = 0. \end{aligned}$$

A similar argument holds for subcases 2, 3, and 4. Hence, our result holds for case 1.

• **Case 2:**  $\downarrow v_1$  is produced by  $m'_i(\downarrow v_1 \otimes (\downarrow w)^{\otimes (i-2)} \otimes \downarrow v_2)$ .

We examine the 2 different possibilities for which  $m'_j$  can yield this arrangement:

- (i)  $m'_j$  produces the  $\downarrow v_1$ .
- (ii)  $m'_j$  produces a  $\downarrow w$  in  $(\downarrow w)^{\otimes (i-2)}$ .

A similar observation to case 1 can be made here regarding the original arrangement  $\downarrow x_1 \otimes \cdots \otimes \downarrow x_n$  containing exactly 3  $v$ 's, once again with  $x_1 = v_1$ . In this case,  $x_n = v_2$ . This yields 2 subcases:

- *Subcase 1:* We have  $m'_i m'_j (\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes (n-k-3)} \otimes \downarrow v_2)$
- *Subcase 2:* We have  $m'_i m'_j (\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_2 \otimes (\downarrow w)^{\otimes (n-k-3)} \otimes \downarrow v_2)$

Let us consider subcase 1:

(i)  $m'_j$  must take both  $\downarrow v_1$ 's. We have:

$$m'_i \left( \left[ m'_j (\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes j-k-2}) \right] \otimes (\downarrow w)^{\otimes n-j-1} \otimes \downarrow v_2 \right) = \downarrow v_1$$

Now  $k + 2 \leq j \leq n - 1$ , so there are  $(n - 1) - (k + 2) + 1 = n - k - 2$  such terms.

(ii)  $m'_j$  must take the second  $\downarrow v_1$  only. We have:

$$(-1)^{|v_1|+k|w|-(k+1)} m'_i \left( \downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \left[ m'_j (\downarrow v_1 \otimes (\downarrow w)^{\otimes j-1}) \otimes (\downarrow w)^{\otimes n-k-j-2} \right] \otimes \downarrow v_2 \right) = - \downarrow v_1$$

Now  $1 \leq j \leq n - k - 2$ , so there are  $(n - k - 2) - (1) + 1 = n - k - 2$  such terms.

This implies that

$$\sum_{i+j=n+1} m'_i m'_j (\downarrow v_1 \otimes (\downarrow w)^{\otimes k} \otimes \downarrow v_1 \otimes (\downarrow w)^{\otimes (n-k-3)} \otimes \downarrow v_2) = (n - k - 2) \downarrow v_1 - (n - k - 2) \downarrow v_1 = 0.$$

A similar argument may be made for subcase 2. Hence, our result holds for case 2.

$$\text{So } \sum_{i+j=n+1} m'_i m'_j (\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) = 0, \forall x_1 \cdots x_n \in V.$$

$$\text{Thus } D^2(\downarrow x_1 \otimes \cdots \otimes \downarrow x_n) = 0$$

By induction,  $D^2 = 0$  on any number of inputs.

Hence the preceding maps  $m_n$  defined on the graded vector space  $V$  form an  $A_\infty$  algebra structure.  $\square$

### 3. INDUCED $L_\infty$ ALGEBRA

The  $A_\infty$  algebra structure on  $V = V_0 \oplus V_1$  that was constructed in this note can be skew symmetrized to yield an  $L_\infty$  algebra structure on  $V$ ; see Theorem 3.1 in [5] for details. This  $L_\infty$  algebra will thus join the collection of previously defined such structures on  $V$ . The relationship among these algebras will be a topic for future research.

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DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH NC 27695, USA

*E-mail address:* `mpallocc@math.ncsu.edu`

DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH NC 27695, USA

*E-mail address:* `lada@math.ncsu.edu`